2.2 The Inverse of a Matrix

The inverse of a real number $a$ is denoted by $a^{-1}$. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix $A$ is said to be invertible if there is an $n \times n$ matrix $C$ satisfying

$$CA = AC = I_n$$

where $I_n$ is the $n \times n$ identity matrix. We call $C$ the inverse of $A$.

**FACT** If $A$ is invertible, then the inverse is unique.

*Proof:* Assume $B$ and $C$ are both inverses of $A$. Then

$$B = BI = B(\text{____}) = (\text{____})\text{____} = I\text{____} = C.$$  

So the inverse is unique since any two inverses coincide. ■

The inverse of $A$ is usually denoted by $A^{-1}$.

We have

$$AA^{-1} = A^{-1}A = I_n$$

**Not all $n \times n$ matrices are invertible.** A matrix which is not invertible is sometimes called a singular matrix. An invertible matrix is called nonsingular matrix.
Theorem 4
Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). If \( ad - bc \neq 0 \), then \( A \) is invertible and
\[
A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]
If \( ad - bc = 0 \), then \( A \) is not invertible.

Assume \( A \) is any invertible matrix and we wish to solve \( Ax = b \). Then
\[
______Ax = _____b \quad \text{and so}
\]
\[
Ix = _______ \text{ or } x = _______.
\]
Suppose \( w \) is also a solution to \( Ax = b \). Then \( Aw = b \) and
\[
______Aw = _____b \quad \text{which means} \quad w = A^{-1}b.
\]
So, \( w = A^{-1}b \), which is in fact the same solution.

We have proved the following result:

Theorem 5
If \( A \) is an invertible \( n \times n \) matrix, then for each \( b \) in \( \mathbb{R}^n \), the equation \( Ax = b \) has the unique solution \( x = A^{-1}b \).
**EXAMPLE:** Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$
-7x_1 + 3x_2 = 2 \\
5x_1 - 2x_2 = 1
$$

*Solution:* Matrix form of the linear system:

$$
\begin{bmatrix}
-7 & 3 \\
5 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
1
\end{bmatrix}
$$

$$
A^{-1} = \frac{1}{14-15}
\begin{bmatrix}
-2 & -3 \\
-5 & -7
\end{bmatrix}
= 
\begin{bmatrix}
2 & 3 \\
5 & 7
\end{bmatrix}
$$

$$
x = A^{-1}b = 
\begin{bmatrix}
2 & 3 \\
5 & 7
\end{bmatrix}
\begin{bmatrix}
\end{bmatrix}
= 
\begin{bmatrix}
\end{bmatrix}
$$
Theorem 6  Suppose $A$ and $B$ are invertible. Then the following results hold:

a. $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$ (i.e. $A$ is the inverse of $A^{-1}$).

b. $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$

Partial proof of part b:

$$(AB)(B^{-1}A^{-1}) = A(\underline{\quad})A^{-1}$$

$$= A(\underline{\quad})A^{-1} = \underline{\quad} = \underline{\quad}.$$ 

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

Theorem 6, part b can be generalized to three or more invertible matrices:

$$(ABC)^{-1} = \underline{\quad}$$

Earlier, we saw a formula for finding the inverse of a $2 \times 2$ invertible matrix. How do we find the inverse of an invertible $n \times n$ matrix? To answer this question, we first look at elementary matrices.
Elementary Matrices

Definition

An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

EXAMPLE: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

$E_1$, $E_2$, and $E_3$ are elementary matrices. Why?
Observe the following products and describe how these products can be obtained by elementary row operations on $A$.

\[
E_1A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
= \begin{bmatrix}
a & b & c \\
2d & 2e & 2f \\
g & h & i
\end{bmatrix}
\]

\[
E_2A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
= \begin{bmatrix}
a & b & c \\
g & h & i \\
d & e & f
\end{bmatrix}
\]

\[
E_3A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
3 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
= \begin{bmatrix}
a & b & c \\
d & e & f \\
3a + g & 3b + h & 3c + i
\end{bmatrix}
\]

*If an elementary row operation is performed on an $m \times n$ matrix $A$, the resulting matrix can be written as $EA$, where the $m \times m$ matrix $E$ is created by performing the same row operations on $I_m$.**
Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix $E$, determine the elementary row operation needed to transform $E$ back into $I$ and apply this operation to $I$ to find the inverse.

For example,

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} \end{bmatrix}$$
Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3(E_2E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$E_3E_2E_1A = I_3.$$

Then multiplying on the right by $A^{-1}$, we get

$$E_3E_2E_1A A^{-1} = I_3 A^{-1}.$$

So

$$E_3E_2E_1I_3 = A^{-1}.$$
The elementary row operations that row reduce $A$ to $I_n$ are the same elementary row operations that transform $I_n$ into $A^{-1}$.

**Theorem 7**

An $n \times n$ matrix $A$ is invertible if and only if $A$ is row equivalent to $I_n$, and in this case, any sequence of elementary row operations that reduces $A$ to $I_n$ will also transform $I_n$ to $A^{-1}$.

**Algorithm for finding $A^{-1}$**

Place $A$ and $I$ side-by-side to form an augmented matrix $[A \ I]$. Then perform row operations on this matrix (which will produce identical operations on $A$ and $I$). So by Theorem 7:

$$[A \ I] \text{ will row reduce to } [I \ A^{-1}]$$

or $A$ is not invertible.

**EXAMPLE:** Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

**Solution:**

$$[A \ I] = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$
Order of multiplication is important!

**EXAMPLE**  Suppose \( A, B, C, \) and \( D \) are invertible \( n \times n \) matrices and \( A = B(D - I_n)C \).

Solve for \( D \) in terms of \( A, B, C \) and \( D \).

**Solution:**

\[
\begin{align*}
    \quad A \quad & = \quad B(D - I_n)C \\
    D - I_n & = B^{-1}A C^{-1} \\
    D - I_n + \quad & = B^{-1}A C^{-1} + \quad \\
    D & = \quad \quad \quad \quad \\
\end{align*}
\]