2.3 Characterizations of Invertible Matrices

Theorem 8 (The Invertible Matrix Theorem)

Let $A$ be a square $n \times n$ matrix. The following statements are equivalent (i.e., for a given $A$, they are either all true or all false).

a. $A$ is an invertible matrix.
b. $A$ is row equivalent to $I_n$.
c. $A$ has $n$ pivot positions.
d. The equation $Ax = 0$ has only the trivial solution.
e. The columns of $A$ form a linearly independent set.
f. The linear transformation $x \rightarrow Ax$ is one-to-one.
g. The equation $Ax = b$ has at least one solution for each $b$ in $\mathbb{R}^n$.
h. The columns of $A$ span $\mathbb{R}^n$.
i. The linear transformation $x \rightarrow Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix $C$ such that $CA = I_n$.
k. There is an $n \times n$ matrix $D$ such that $AD = I_n$.
l. $A^T$ is an invertible matrix.
EXAMPLE: Use the Invertible Matrix Theorem to determine if $A$ is invertible, where

$$
A = \begin{bmatrix}
1 & -3 & 0 \\
-4 & 11 & 1 \\
2 & 7 & 3
\end{bmatrix}.
$$

\textit{Solution}

\[
A = \begin{bmatrix}
1 & -3 & 0 \\
-4 & 11 & 1 \\
2 & 7 & 3
\end{bmatrix} \sim \cdots \sim \left[
\begin{array}{ccc}
1 & -3 & 0 \\
0 & -1 & 1 \\
0 & 0 & 16
\end{array}
\right]
\]

3 pivots positions

\textit{Circle correct conclusion:} Matrix $A$ \textit{is} / \textit{is not} invertible.
**EXAMPLE:** Suppose $H$ is a $5 \times 5$ matrix and suppose there is a vector $v$ in $\mathbb{R}^5$ which is not a linear combination of the columns of $H$. What can you say about the number of solutions to $Hx = 0$?

*Solution*  
Since $v$ in $\mathbb{R}^5$ is not a linear combination of the columns of $H$, the columns of $H$ do not \__________ $\mathbb{R}^5$.

So by the Invertible Matrix Theorem, $Hx = 0$ has

\______________________________.
Invertible Linear Transformations

For an invertible matrix $A$, 

$$A^{-1}Ax = x \text{ for all } x \text{ in } \mathbb{R}^n$$

and

$$AA^{-1}x = x \text{ for all } x \text{ in } \mathbb{R}^n.$$ 

Pictures:
A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(x)) = x \text{ for all } x \text{ in } \mathbb{R}^n$$

and

$$T(S(x)) = x \text{ for all } x \text{ in } \mathbb{R}^n.$$ 

**Theorem 9**

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $A$ be the standard matrix for $T$. Then $T$ is invertible if and only if $A$ is an invertible matrix. In that case, the linear transformation $S$ given by $S(x) = A^{-1}x$ is the unique function satisfying

$$S(T(x)) = x \text{ for all } x \text{ in } \mathbb{R}^n$$

and

$$T(S(x)) = x \text{ for all } x \text{ in } \mathbb{R}^n.$$