4.5 The Dimension of a Vector Space

THEOREM 9

If a vector space $V$ has a basis $\beta = \{b_1, \ldots, b_n\}$, then any set in $V$ containing more than $n$ vectors must be linearly dependent.

**Proof:** Suppose $\{u_1, \ldots, u_p\}$ is a set of vectors in $V$ where $p > n$. Then the coordinate vectors $\{[u_1]_{\beta}, \ldots, [u_p]_{\beta}\}$ are in $\mathbb{R}^n$. Since $p > n$, $\{[u_1]_{\beta}, \ldots, [u_p]_{\beta}\}$ are linearly dependent and therefore $\{u_1, \ldots, u_p\}$ are linearly dependent. $\blacksquare$

THEOREM 10

If a vector space $V$ has a basis of $n$ vectors, then every basis of $V$ must consist of $n$ vectors.

**Proof:** Suppose $\beta_1$ is a basis for $V$ consisting of exactly $n$ vectors. Now suppose $\beta_2$ is any other basis for $V$. By the definition of a basis, we know that $\beta_1$ and $\beta_2$ are both linearly independent sets.

By Theorem 9, if $\beta_1$ has more vectors than $\beta_2$, then ______ is a linearly dependent set (which cannot be the case).

Again by Theorem 9, if $\beta_2$ has more vectors than $\beta_1$, then ______ is a linearly dependent set (which cannot be the case).

Therefore $\beta_2$ has exactly $n$ vectors also. $\blacksquare$
DEFINITION

If $V$ is spanned by a finite set, then $V$ is said to be **finite-dimensional**, and the **dimension** of $V$, written as $\dim V$, is the number of vectors in a basis for $V$. The dimension of the zero vector space $\{0\}$ is defined to be 0. If $V$ is not spanned by a finite set, then $V$ is said to be **infinite-dimensional**.

EXAMPLE: The standard basis for $P_3$ is $\{\}$.

$\dim P_3 = \_\_\_\_\_.$

In general, $\dim P_n = n + 1$.

EXAMPLE: The standard basis for $R^n$ is $\{e_1, \ldots, e_n\}$ where $e_1, \ldots, e_n$ are the columns of $I_n$. So, for example, $\dim R^3 = 3$. 

2
EXAMPLE: Find a basis and the dimension of the subspace

\[ W = \left\{ \begin{bmatrix} a + b + 2c \\ 2a + 2b + 4c + d \\ b + c + d \\ 3a + 3c + d \end{bmatrix} : a, b, c, d \text{ are real} \right\}. \]

Solution: Since

\[
\begin{bmatrix} a + b + 2c \\ 2a + 2b + 4c + d \\ b + c + d \\ 3a + 3c + d \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},
\]

\[ W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}, \]

where \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \]

- Note that \( \mathbf{v}_3 \) is a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \), so by the Spanning Set Theorem, we may discard \( \mathbf{v}_3 \).

- \( \mathbf{v}_4 \) is not a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). So \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\} \) is a basis for \( W \). Also, \( \dim W = \_\).
EXAMPLE: *Dimensions of subspaces of $\mathbb{R}^3$*

0-*dimensional subspace* contains only the zero vector
\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}.
\]

1-*dimensional subspaces.* $\text{Span}\{v\}$ where $v \neq 0$ is in $\mathbb{R}^3$.

These subspaces are ______________ through the origin.

2-*dimensional subspaces.* $\text{Span}\{u, v\}$ where $u$ and $v$ are in $\mathbb{R}^3$ and are not multiples of each other.

These subspaces are ______________ through the origin.

3-*dimensional subspaces.* $\text{Span}\{u, v, w\}$ where $u$, $v$, $w$ are linearly independent vectors in $\mathbb{R}^3$. This subspace is $\mathbb{R}^3$ itself because the columns of $A = \begin{bmatrix} u & v & w \end{bmatrix}$ span $\mathbb{R}^3$ according to the IMT.
THEOREM 11

Let $H$ be a subspace of a finite-dimensional vector space $V$. Any linearly independent set in $H$ can be expanded, if necessary, to a basis for $H$. Also, $H$ is finite-dimensional and \[ \dim H \leq \dim V. \]

EXAMPLE: Let $H = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Then $H$ is a subspace of $\mathbb{R}^3$ and $\dim H < \dim \mathbb{R}^3$.

We could expand the spanning set $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to form a basis for $\mathbb{R}^3$. 

THEOREM 12  THE BASIS THEOREM

Let $V$ be a $p$–dimensional vector space, $p \geq 1$. Any linearly
independent set of exactly $p$ vectors in $V$ is automatically a
basis for $V$. Any set of exactly $p$ vectors that spans $V$ is
automatically a basis for $V$.

EXAMPLE: Show that $\{t, 1 - t, 1 + t - t^2\}$ is a basis for $P_2$.

Solution: Let $v_1 = t, v_2 = 1 - t, v_3 = 1 + t - t^2$ and $\beta = \{1, t, t^2\}$.

Corresponding coordinate vectors

$$[v_1]_{\beta} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [v_2]_{\beta} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, [v_3]_{\beta} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$[v_2]_{\beta}$ is not a multiple of $[v_1]_{\beta}$

$[v_3]_{\beta}$ is not a linear combination of $[v_1]_{\beta}$ and $[v_2]_{\beta}$

$\Rightarrow \{[v_1]_{\beta}, [v_2]_{\beta}, [v_3]_{\beta}\}$ is linearly independent and therefore
$\{v_1, v_2, v_3\}$ is also linearly independent.

Since $\dim P_2 = 3$, $\{v_1, v_2, v_3\}$ is a basis for $P_2$ according to The
Basis Theorem.
Dimensions of Col $A$ and Nul $A$

Recall our techniques to find basis sets for column spaces and null spaces.

**EXAMPLE:** Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$. Find dim Col $A$ and dim Nul $A$.

**Solution**

\[
\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]

So \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \\ 8 \end{bmatrix} \right\} \) is a basis for Col $A$ and $\text{dim Col } A = 2$. 
Now solve $Ax = 0$ by row-reducing the corresponding augmented matrix. Then we arrive at

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 0 \\
2 & 4 & 7 & 8 & 0
\end{bmatrix}
\sim \ldots \sim
\begin{bmatrix}
1 & 2 & 0 & 4 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
$$

$x_1 = -2x_2 - 4x_4$

$x_3 = 0$

and

$$
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
\phantom{x} \\
\phantom{x} \\
\phantom{x}
\end{bmatrix}
+ 
\begin{bmatrix}
-2 \\
\phantom{x} \\
\phantom{x} \\
\phantom{x}
\end{bmatrix}
+ 
\begin{bmatrix}
-4 \\
\phantom{x} \\
\phantom{x} \\
\phantom{x}
\end{bmatrix}
$$

So $$\left\{ \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-4 \\
0 \\
0 \\
1
\end{bmatrix} \right\}$$ is a basis for $\text{Nul } A$ and

$\text{dim Nul } A = 2$.

Note

$$
\begin{align*}
\text{dim Col } A &= \text{number of pivot columns of } A \\
\text{dim Nul } A &= \text{number of free variables of } A
\end{align*}
$$