Section 4.6 Rank

The set of all linear combinations of the row vectors of a matrix \( A \) is called the row space of \( A \) and is denoted by Row \( A \).

**EXAMPLE:** Let

\[
A = \begin{bmatrix}
-1 & 2 & 3 & 6 \\
2 & -5 & -6 & -12 \\
1 & -3 & -3 & -6
\end{bmatrix}
\]

and

\[
r_1 = (-1, 2, 3, 6) \\
r_2 = (2,-5,-6,-12) \\
r_3 = (1,-3,-3,-6)
\]

Row \( A = \text{Span}\{r_1, r_2, r_3\} \) (a subspace of \( \mathbb{R}^4 \))

While it is natural to express row vectors horizontally, they can also be written as column vectors if it is more convenient. Therefore

\[
\text{Col} \ A^T = \text{Row} \ A.
\]

When we use row operations to reduce matrix \( A \) to matrix \( B \), we are taking linear combinations of the rows of \( A \) to come up with \( B \). We could reverse this process and use row operations on \( B \) to get back to \( A \). Because of this, the row space of \( A \) equals the row space of \( B \).

**THEOREM 13**

If two matrices \( A \) and \( B \) are row equivalent, then their row spaces are the same. If \( B \) is in echelon form, the nonzero rows of \( B \) form a basis for the row space of \( A \) as well as \( B \).
EXAMPLE: The matrices

\[ A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \]  
\[ B = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

are row equivalent. Find a basis for row space, column space and null space of \( A \). Also state the dimension of each.

Basis for Row \( A \) :

\[ \{ \} \]

\[ \text{dim Row } A : \underline{\_ \_ \_} \]

Basis for Col \( A \) :

\[ \left\{ \begin{bmatrix} \_ \\ \_ \\ \_ \end{bmatrix}, \begin{bmatrix} \_ \\ \_ \end{bmatrix} \right\} \]

\[ \text{dim Col } A : \underline{\_ \_ \_} \]
To find Nul $A$, solve $Ax = 0$ first:

$$
\begin{bmatrix}
-1 & 2 & 3 & 6 & 0 \\
2 & -5 & -6 & -12 & 0 \\
1 & -3 & -3 & -6 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
-1 & 2 & 3 & 6 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}

\sim
\begin{bmatrix}
1 & 0 & -3 & -6 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}

\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
3x_3 + 6x_4 \\
0 \\
x_3 \\
x_4
\end{bmatrix}
= x_3 \begin{bmatrix}
3 \\
0 \\
1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}

Basis for Nul $A$: \{ \begin{bmatrix}
3 \\
0 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
6 \\
0 \\
0 \\
1
\end{bmatrix} \}

and dim Nul $A = _______
Note the following:
\[\dim \text{Col } A = \# \text{ of pivots of } A = \# \text{ of nonzero rows in } B = \dim \text{Row } A.\]

\[\dim \text{Nul } A = \# \text{ of free variables} = \# \text{ of nonpivot columns of } A.\]

**DEFINITION**
The rank of \( A \) is the dimension of the column space of \( A \).

\[\text{rank } A = \dim \text{Col } A = \# \text{ of pivot columns of } A = \dim \text{Row } A.\]

\[
\begin{align*}
\text{rank } A &+ \dim \text{Nul } A = n \\
\updownarrow & \quad \updownarrow \\
\# \text{ of pivot columns of } A &+ \# \text{ of nonpivot columns of } A
\end{align*}
\]

**THEOREM 14 THE RANK THEOREM**
The dimensions of the column space and the row space of an \( m \times n \) matrix \( A \) are equal. This common dimension, the rank of \( A \), also equals the number of pivot positions in \( A \) and satisfies the equation

\[\text{rank } A + \dim \text{Nul } A = n.\]
Since Row $A = \text{Col } A^T$, 

$$\text{rank } A = \text{rank } A^T.$$ 

**EXAMPLE:** Suppose that a $5 \times 8$ matrix $A$ has rank 5. Find dim Nul $A$, dim Row $A$ and rank $A^T$. Is Col $A = \mathbb{R}^5$? 

**Solution:** 

$$\begin{align*}
\underline{\text{rank } A} + \underline{\text{dim Nul } A} &= \underline{n} \\
5 &+ \ ? &= 8
\end{align*}$$

$$5 + \dim \text{Nul } A = 8 \quad \Rightarrow \quad \dim \text{Nul } A = \____$$

$$\dim \text{Row } A = \text{rank } A = \____$$

$$\Rightarrow \quad \text{rank } A^T = \text{rank } \____ = \____$$

Since rank $A = \#$ of pivots in $A = 5$, there is a pivot in every row. So the columns of $A$ span $\mathbb{R}^5$ (by Theorem 4, page 43). Hence Col $A = \mathbb{R}^5$. 
EXAMPLE: For a $9 \times 12$ matrix $A$, find the smallest possible value of $\dim \text{Nul} \ A$.

Solution:

$$\text{rank} \ A + \dim \text{Nul} \ A = 12$$

$$\dim \text{Nul} \ A = 12 - \text{rank} \ A$$

largest possible value = _____

smallest possible value of $\dim \text{Nul} \ A = _____$
Visualizing Row $A$ and Nul $A$

**EXAMPLE:** Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$. One can easily verify the following:

Basis for Nul $A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and therefore Nul $A$ is a plane in $\mathbb{R}^3$.

Basis for Row $A = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and therefore Row $A$ is a line in $\mathbb{R}^3$.

Basis for Col $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and therefore Col $A$ is a line in $\mathbb{R}^2$.

Basis for Nul $A^T = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and therefore Nul $A^T$ is a line in $\mathbb{R}^2$. 
Subspaces Nul $A$ and Row $A$

Subspaces Nul $A^T$ and Col $A$
The Rank Theorem provides us with a powerful tool for determining information about a system of equations.

**EXAMPLE:** A scientist solves a homogeneous system of 50 equations in 54 variables and finds that exactly 4 of the unknowns are free variables. Can the scientist be *certain* that any associated nonhomogeneous system (with the same coefficients) has a solution?

**Solution:** Recall that

\[
\text{rank } A = \dim \text{ Col } A = \# \text{ of pivot columns of } A
\]

\[
\dim \text{ Nul } A = \# \text{ of free variables}
\]

In this case \(Ax = 0\) of where \(A\) is \(50 \times 54\).

By the rank theorem,

\[
\text{rank } A + \underline{\text{_______}} = \underline{\text{_______}}
\]

or

\[
\text{rank } A = \underline{\text{_______}}.
\]

So any nonhomogeneous system \(Ax = b\) has a solution because there is a pivot in every row.
THE INVERTIBLE MATRIX THEOREM (continued)

Let $A$ be a square $n \times n$ matrix. The following statements are equivalent:

m. The columns of $A$ form a basis for $\mathbb{R}^n$

n. $\text{Col } A = \mathbb{R}^n$

o. $\dim \text{Col } A = n$

p. $\text{rank } A = n$

q. $\text{Nul } A = \{0\}$

r. $\dim \text{Nul } A = 0$